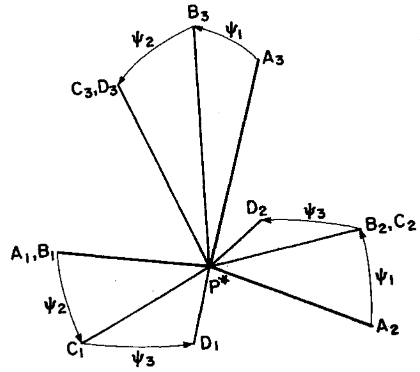
Fig. 3 Attitude angles



conclusion that the corresponding motion is stable.‡

Stabilization can be accomplished with relatively light particles performing oscillations of relatively small amplitude and low frequency, all of which points in the direction of practical feasibility. For example, if R is a homogeneous, right-circular cylinder of mass M, with equal radius and height of 5 ft, then R can be stabilized (for motion with $\omega/\Omega = -1$) by particles of mass $m = 0.012M$ oscillating with an amplitude $B = 10$ in. and with a circular frequency $p = 10\Omega$.

‡ The necessary computations were performed on a Burroughs 220 computer, generously made available to the authors by the Computation Center of Stanford University.

Conclusion

It has been shown that attitude stabilization by means of the proposed mechanism is possible in principle and may be feasible in practice. Actual design, and particularly optimum design, of a satellite stabilized in this fashion would require repeated use of the procedure described earlier and would be facilitated by prior exploration of the five-dimensional parameter space of the problem. Such an exploration is in progress and may be the subject of a later paper.

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Optimal Low-Thrust Near-Circular Orbital Transfer

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Optimum low-thrust transfers between neighboring coplanar circular orbits have been studied for a simplified system model. Small deviations from an original circular orbit motion are assumed, and the thrust acceleration is taken to be constant. Within the limits of these assumptions, the numerical results are sufficient to describe in general the optimum thrust steering program and performance capability of a vehicle in terms of its thrust/weight ratio, orbital frequency, and thrust duration. For values of thrust duration equal to an integral multiple of the orbital period, the optimum thrust direction is continuously circumferential.

Nomenclature

$a_{1,2,3,4}$ = functions of τ defined by Eqs. (17)
A,B = functions of τ defined by Eqs. (32) and (33)
$b_{1,2,3,4}$ = functions of τ defined by Eqs. (17)
C = $(A^2 + B^2)^{1/2}$
F = integrand of I
I = functional to be minimized
ΔI = total variation of I from its minimum value
J_1,J_2 = integrals required to vanish for terminal conditions satisfying circular orbital motion
m = mass of the vehicle
P = const = $T/m\omega_0^2$
r = radial distance of the vehicle from the center of the earth
t = time
T = thrust of the vehicle
u = $dx/d\tau$
v = $dy/d\tau$
x,y = position components of the vehicle defined in Fig. 1
θ = thrust steering angle defined in Fig. 1
$\delta\theta$ = variation of θ from its optimum value
Λ_1,Λ_2 = constant Lagrange multipliers
μ = gravitational parameter of the earth
τ = nondimensional time = $\omega_0 t$
τ_1 = arbitrary nondimensional time, $0 \leq \tau_1 \leq \tau$
ω = circular orbital frequency of the vehicle

Subscripts

0 = initial values at $\tau = 0$
f = final values at $\tau = \tau_f$

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Introduction

The problem of determining optimum rocket trajectories by the indirect methods of the calculus of variations has received considerable attention in the past decade. The well-known bilinear tangent thrust steering program for optimum boost maneuvers in a vacuum and a constant gravitational field has been covered amply in the literature.[1–9]

Another space operation of increasing interest is the determination of optimal low-thrust orbital maneuvers. In order to understand this problem better and to gain insight into the character of optimum thrust steering programs, an analysis is carried out in this paper for a simplified system model. Previous treatments of this problem employing simplifying assumptions are described in Refs. 10–12. The basic assumptions and some of the results presented here are quite similar to those of Ref. 10. However, the derivation of the optimal steering program in this paper, as distinguished from those of others, employs integral instead of differential side conditions. This is possible because the equations of motion first are linearized. By using the equations of motion employed by Clohessy and Wiltshire in a satellite rendezvous analysis[13] and by assuming that the thrust/mass ratio is constant, the equations become linear with constant coefficients.

Also of possible interest in this paper are the results obtained by integrating the equations of motion numerically. These results give a complete survey of the optimum trajectories for maneuvers completed in fractions of an orbital period to those requiring several revolutions. Also shown is the transition to the solution for many revolutions.

Equations of Motion

The equations of motion employed for the analysis, like those of Ref. 13, are expressed in coordinates of a rotating axis system with the origin O moving in a circular orbit about the earth, as shown in Fig. 1. The positive y axis points away from the gravitational center, and the x axis is oriented along the tangent to the circular orbit with the positive direction opposite to that of the orbital velocity vector. Only the planar motion case is analyzed. If the displacement of the vehicle from the origin is assumed small, the equations simplify to the following on linearization:

$$d^2x/dt^2 = (T/m)\cos\theta + 2\omega_0\, dy/dt \tag{1}$$

$$d^2y/dt^2 = (T/m)\sin\theta - 2\omega_0\, dx/dt + 3\omega_0{}^2 y \tag{2}$$

Here T is the thrust, m the mass of the vehicle, θ the thrust steering angle (Fig. 1), and ω_0 the angular frequency of the origin's circular orbit motion.

Equations (1) and (2) simplify further by transformation to a new independent variable,

$$\tau = \omega_0 t \tag{3}$$

The variable τ is nondimensional and may be interpreted physically as the orbital angle between the rotating origin and some inertial reference (Fig. 1). The equations of motion may be written in first-order form as

$$du/d\tau = P\cos\theta + 2v \tag{4}$$

$$dv/d\tau = P\sin\theta - 2u + 3y \tag{5}$$

$$dx/d\tau = u \tag{6}$$

$$dy/d\tau = v \tag{7}$$

$$P = T/m\omega_0{}^2 \tag{8}$$

If P is assumed constant and θ is a function of τ, then the

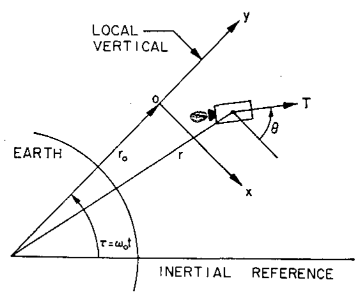

Fig. 1 Orbital transfer schematic for linear low-thrust analysis

solution of Eqs. (4–7) is

$$u(\tau_1) = 2[2u_0 - 3y_0]\cos\tau_1 + 2v_0\sin\tau_1 + 6y_0 - 3u_0 + P\int_0^{\tau_1}\{[4\cos(\tau_1-\tau) - 3]\cos\theta + 2\sin(\tau_1-\tau)\sin\theta\}\,d\tau \tag{9}$$

$$v(\tau_1) = v_0\cos\tau_1 + [3y_0 - 2u_0]\sin\tau_1 + P\int_0^{\tau_1}\{-2\sin(\tau_1-\tau)\cos\theta + \cos(\tau_1-\tau)\sin\theta\}\,d\tau \tag{10}$$

$$x(\tau_1) = 2(2u_0 - 3y_0)\sin\tau_1 - 2v_0\cos\tau_1 + (6y_0 - 3u_0)\tau_1 + (x_0 + 2v_0) + P\int_0^{\tau_1}\{[4\sin(\tau_1-\tau) - 3(\tau_1-\tau)]\cos\theta + 2[1-\cos(\tau_1-\tau)]\sin\theta\}\,d\tau \tag{11}$$

$$y(\tau_1) = (2u_0 - 3y_0)\cos\tau_1 + v_0\sin\tau_1 + 4y_0 - 2u_0 + P\int_0^{\tau_1}\{2[\cos(\tau_1-\tau) - 1]\cos\theta + \sin(\tau_1-\tau)\sin\theta\}\,d\tau \tag{12}$$

where $0 \leq \tau_1 \leq \tau_f$ and u_0,v_0,x_0,y_0 are initial conditions. The freefall ($P = 0$) part of the foregoing solution was introduced first by Clohessy and Wiltshire[13] and Wheelon.[14]

Prior to maneuvering, the vehicle is considered to be in a circular orbit and coincident with the origin of the reference system. Hence, all initial conditions are zero. The equations for the final velocity and position components reduce to

$$u_f = P\int_0^{\tau_f}\{a_1\cos\theta + b_1\sin\theta\}\,d\tau \tag{13}$$

$$v_f = P\int_0^{\tau_f}\{a_2\cos\theta + b_2\sin\theta\}\,d\tau \tag{14}$$

$$x_f = P\int_0^{\tau_f}\{a_3\cos\theta + b_3\sin\theta\}\,d\tau \tag{15}$$

$$y_f = P\int_0^{\tau_f}\{a_4\cos\theta + b_4\sin\theta\}\,d\tau \tag{16}$$

where

$$\begin{aligned} a_1 &= 4\cos(\tau_f-\tau) - 3 \\ a_2 &= -2\sin(\tau_f-\tau) \\ a_3 &= 4\sin(\tau_f-\tau) - 3(\tau_f-\tau) \\ a_4 &= 2\cos(\tau_f-\tau) - 2 \\ b_1 &= 2\sin(\tau_f-\tau) \\ b_2 &= \cos(\tau_f-\tau) \\ b_3 &= 2 - 2\cos(\tau_f-\tau) \\ b_4 &= \sin(\tau_f-\tau) \end{aligned} \tag{17}$$

Terminal Conditions

The final velocity components required to establish a circular orbit first may be derived without resort to simplifying assumptions. These components, as measured in the rotating and translating reference frame, are

$$(dx/dt)_f = -(r_0 + y_f)(\omega_f - \omega_0) \tag{18}$$

$$(dy/dt)_f = x(\omega_f - \omega_0) \tag{19}$$

where r_0 is the initial radial distance of the vehicle from the center of the earth (Fig. 1), and ω_f is the vehicle's final circular orbital frequency given by

$$\omega_f{}^2 = \mu/r_f{}^3 \tag{20}$$

where the constant μ is the earth's gravitational parameter and

$$r_f{}^2 = x_f{}^2 + (r_0 + y_f)^2 \tag{21}$$

Since the term $(\omega_f - \omega_0)$ is the change in orbital frequency,

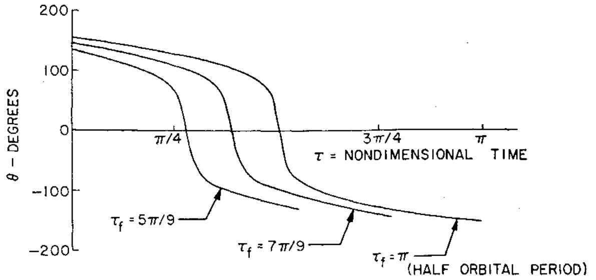

Fig. 2 Optimum thrust direction time histories for transfer times up to one-half orbital period

which is assumed to be small, an approximation may be derived by differentiating Eqs. (20) and (21), replacing the differentials by finite differences,

$$d\omega \approx \Delta\omega = \omega_f - \omega_0$$
$$dx \approx \Delta x = x$$
$$dy \approx \Delta y = y$$

and combining the result with Eqs. (18) and (19). Hence

$$\left(\frac{dx}{dt}\right)_f = \frac{3\omega_0(r_0 + y_f)}{2r_0r_f}\left[x_f^2 + y_f(r_0 + y_f)\right] \qquad (22)$$

$$\left(\frac{dy}{dt}\right)_f = -\frac{3\omega_0}{2r_0r_f}\left[x_f^3 + x_fy_f(r_0 + y_f)\right] \qquad (23)$$

These equations then may be linearized based on the assumption that both x_f and y_f are small relative to r_0 and r_f. This assumption is the same as the one employed to derive the linearized equations of motion.[13, 14] The terminal conditions simplify to

$$(dx/dt)_f = \tfrac{3}{2}\,\omega_0 y_f \qquad (24)$$

$$(dy/dt)_f = 0 \qquad (25)$$

Transformation of the independent variable from t to τ results in

$$u_f = \tfrac{3}{2} y_f \qquad (26)$$

$$v_f = 0 \qquad (27)$$

Variational Treatment

Determination of the optimal maneuvers for circle-to-circle orbital transfer may be effected in several ways. Essentially, the problem is one of maximizing the integral y_f given by Eq. (16) subject to the terminal conditions, Eqs. (26) and (27). Substitution of Eqs. (13, 14, 16, and 17) into Eqs. (26) and (27) results in the following two integral expressions, each required to vanish:

$$J_1 = \int_0^{\tau_f} \left[2\cos(\tau_f - \tau)\cos\theta + \sin(\tau_f - \tau)\sin\theta\right] d\tau = 0 \qquad (28)$$

$$J_2 = \int_0^{\tau_f} \left[\cos(\tau_f - \tau)\sin\theta - 2\sin(\tau_f - \tau)\cos\theta\right] d\tau = 0 \qquad (29)$$

This has the form of the classical isoperimetrical problem and is treated as such. However, because of the convenient absence of the derivative $d\theta/d\tau$ in the integrals, it is advantageous to rederive the Euler-Lagrange equation, the Legendre-Clebsch condition, and other necessary and sufficient conditions.

As in previous derivations, constant Lagrange multipliers are introduced and the integral

$$I = -y_f + \Lambda_1 J_1 + \Lambda_2 J_2 = \int_0^{\tau_f} F(\tau,\theta)\, d\tau \qquad (30)$$

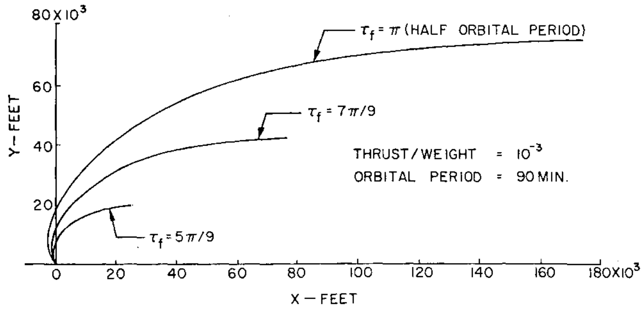

Fig. 3 Optimum trajectories for transfer times up to one-half orbital period

considered, where

$$F(\tau,\theta) = A\sin\theta + B\cos\theta \qquad (31)$$

$$A = (\Lambda_1 - 1)\sin(\tau_f - \tau) + \Lambda_2\cos(\tau_f - \tau) \qquad (32)$$

$$B = 2(\Lambda_1 - 1)\cos(\tau_f - \tau) - 2\Lambda_2\sin(\tau_f - \tau) + 2 \qquad (33)$$

If $\theta(\tau)$ is an admissible function for which I is stationary and $\delta\theta(\tau)$ is an arbitrary variation of $\theta(\tau)$, then the integrand F may be expanded as

$$F(\tau,\theta + \delta\theta) = F(\tau,\theta) + \frac{\partial F(\tau,\theta)}{\partial\theta}\,\delta\theta + \frac{1}{2!}\,\frac{\partial^2 F(\tau,\theta)}{\partial\theta^2}\,(\delta\theta)^2 + \frac{1}{3!}\,\frac{\partial^3 F(\tau,\theta)}{\partial\theta^3}\,(\delta\theta)^3 + \ldots \qquad (34)$$

It follows that the variation of the integral from its stationary value is expressed as

$$\Delta I = \int_0^{\tau_f} \frac{\partial F}{\partial\theta}\,\delta\theta d\tau + \frac{1}{2!}\int_0^{\tau_f} \frac{\partial^2 F}{\partial\theta^2}\,(\delta\theta)^2 d\tau + \frac{1}{3!}\int_0^{\tau_f} \frac{\partial^3 F}{\partial\theta^3}\,(\delta\theta)^3 d\tau + \ldots \qquad (35)$$

To obtain a stationary value of I, the equation

$$\partial F(\tau,\theta)/\partial\theta = 0 \qquad (36)$$

must be satisfied, and to insure that the stationary value is a minimum, the condition

$$\partial^2 F(\tau,\theta)/\partial\theta^2 \geq 0 \qquad (37)$$

also is necessary. These two necessary conditions for a minimum also are derived in Refs. 9 and 15.

The vanishing of the first variation implied by Eq. (36) leads to

$$\tan\theta = A/B \qquad (38)$$

The Legendre-Clebsch condition given by inequality (37) resolves the ambiguity of quadrant for the angle θ, yielding

$$\sin\theta = -A/C \qquad \cos\theta = -B/C \qquad (39)$$

where

$$C = +(A^2 + B^2)^{1/2} \qquad (40)$$

By evaluating all the variations of ΔI beyond the second, it is possible to show that the optimum thrust steering program given by Eq. (39), for those values of Λ_1 and Λ_2 which satisfy Eqs. (28) and (29), will result in an absolute minimum of I for all possible variations of $\delta\theta(\tau)$. Repeated differentiation of F given by Eq. (31) results in

$$\begin{aligned} \partial F/\partial\theta &= 0 & \partial^2 F/\partial\theta^2 &= -F \\ \partial^3 F/\partial\theta^3 &= 0 & \partial^4 F/\partial\theta^4 &= +F \\ \partial^5 F/\partial\theta^5 &= 0 & \partial^6 F/\partial\theta^6 &= -F \end{aligned} \qquad (41)$$

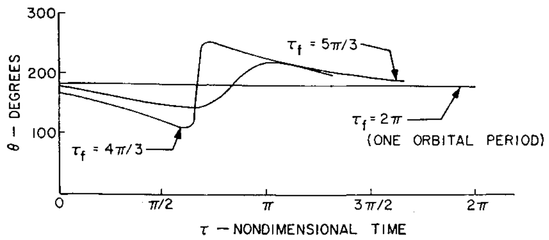

Fig. 4 Optimum thrust direction time histories for transfer times from two-thirds to one orbital period

etc. Substitution of Eqs. (41) into Eq. (35) yields

$$\Delta I = \int_0^{\tau_f} [-F] \left[\frac{1}{2!} (\delta\theta)^2 - \frac{1}{4!} (\delta\theta)^4 + \frac{1}{6!} (\delta\theta)^6 - \ldots \right] d\tau \quad (42)$$

The infinite series in the integrand is recognized as the expansion for $[1 - \cos(\delta\theta)]$. Hence

$$\Delta I = \int_0^{\tau_f} [-F][1 - \cos(\delta\theta)]\, d\tau \quad (43)$$

Since

$$[1 - \cos(\delta\theta)] \geq 0 \quad (44)$$

and

$$\partial^2 F/\partial\theta^2 = -F \geq 0 \quad (45)$$

the total variation ΔI never can be negative, no matter how large a variation of θ is considered. Furthermore, it is positive except for $\delta\theta = 0$ and to integral multiples of $\pm 2\pi$. Mathematically, such variations result in multiple solutions, each with the same minimum value of I. However, physically these solutions are identical, and this is equivalent to Eq. (43) being positive definite. This establishes the unique optimum maneuver given by Eq. (39) as globally minimizing. The only remaining step is the determination of the Lagrange multipliers, Λ_1 and Λ_2.

Determination of Lagrange Multipliers

It is clear that for completion of the analysis the unknown Lagrange multipliers could, in principle, be determined by substitution of Eqs. (32, 33, and 39) into Eqs. (28) and (29) and the integration carried out analytically. The vanishing of these two integrals, J_1 and J_2, for given values of τ_f, would determine Λ_1 and Λ_2 uniquely.

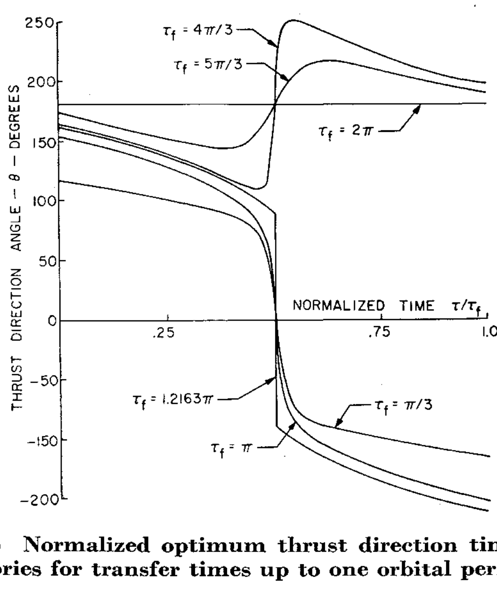

Fig. 5 Normalized optimum thrust direction time histories for transfer times up to one orbital period

Initial attempts to integrate analytically J_1 and J_2 were unsuccessful. Instead, a digital computer was programmed to integrate numerically the equations of motion, generating solutions for prescribed values of Λ_1, Λ_2, and τ_f. A search was made by varying Λ_1 and Λ_2, for a given value of τ_f, until the terminal conditions, Eqs. (26) and (27), were satisfied approximately. The computer then automatically "homed in" on the constants to the desired accuracy. To repeat this procedure for other values of τ_f would have been uneconomical, and hence a perturbation scheme was employed to generate solutions for each succeeding value of τ_f by extrapolating the results of previously iterated solutions. Newton's formula for backward interpolation[16] is ideally suited to this purpose and becomes quite simple for constant τ_f intervals.

Numerical Results and Conclusions

Figures 2–8 summarize the results of 72 iterated solutions covering four orbital periods. An examination of the optimum orbit transfer maneuvers reveals several interesting characteristics.

It is noted (Fig. 2) that the time variation of the thrust steering angle is antisymmetrical with respect to the midpoint. Similarly, time histories of the two components of velocity and acceleration also have this "exactly" symmetrical or antisymmetrical property. "Exactly" refers to an accuracy of six significant figures for the numerically integrated results and suggests that the symmetry property would be exact if the integration were error-free.

Figure 3 shows the trajectories of the vehicle with respect to the rotating axis system for thrust programs shown in Fig. 2. Thrust steering angles for longer transfer times are shown in Fig. 4. Although the same antisymmetrical properties are displayed, it is of interest to note that the θ motion has a mean of $\theta = 180°$ (circumferential thrust), whereas the corresponding motion for the short-term maneuvers (Fig. 2) takes place about a mean of $\theta = 0$ (opposite to circumferential thrust).

This abrupt change in character of the optimum steering program is made more evident in Fig. 5 by rescaling the time of each solution so that a comparison may be made on a common normalized time scale. When this is done, it becomes obvious that between $\tau_f = \pi$ and $\tau_f = 4\pi/3$ the time variation becomes more abrupt. A closer examination reveals that there exists a discontinuous solution at $\tau_f = 1.2163\pi$ for which θ changes instantaneously from 90° to −90° at the midpoint of the maneuver. The instantaneous change also could be interpreted to be from 90° to 270°. This thrust steering "corner" is discussed in further detail by Leitmann[17] and Faulkner.[18] It is the only discontinuous solution encountered for circle-to-circle orbital transfer and is brought about at $\frac{1}{2}\tau_f$ by the simultaneous vanishing of A and B in Eq. (38).

Also shown in Figs. 4 and 5 is the thrust steering angle for $\tau_f = 2\pi$. For this solution, which corresponds to an orbital transfer of the duration of just one revolution, θ remains

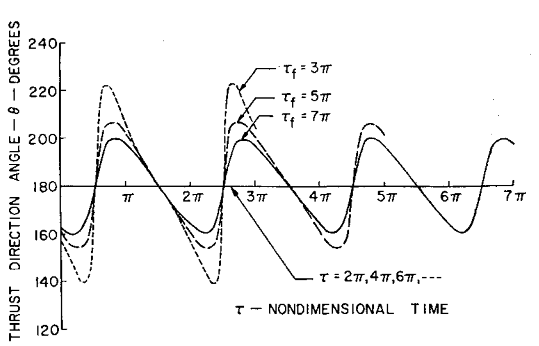

Fig. 6 Optimum thrust direction time histories for transfer times exceeding one orbital period

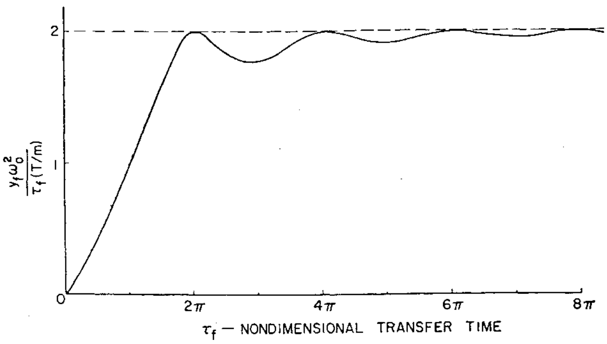

Fig. 7 Variation of nondimensional altitude gain parameter with nondimensional transfer time

constant at 180°. In fact, whenever the duration of powered flight is some integral multiple of the orbital period, the optimum thrust direction is circumferential, and the vehicle passes through a higher energy circular orbit condition at the end of each revolution.

If the circumferential thrust program is inserted in Eqs. (28) and (29), the integrals are evaluated easily, and, indeed, both vanish as expected.

The possibility that the optimal thrust direction program may be circumferential for a circle-to-circle orbital transfer involving many revolutions should offer an exploitation possibility in connection with the guidance problem. Recently, Kelley and Dunn[19] have used this thrust steering program as a nominal maneuver and have applied a second-order guidance approximation scheme for optimum orbital rendezvous in the vicinity of the nominal.

In general, for the longer duration maneuvers, θ oscillates at the orbital frequency. This is shown in Fig. 6 for maneuver durations of $1\frac{1}{2}$, $2\frac{1}{2}$, and $3\frac{1}{2}$ revolutions. It is interesting to note that, as this angle (or final nondimensional time) increases, the amplitude of the oscillation decreases, and a steady-state circumferential thrust program is approached for maneuvers of very long duration.

An interesting result of the numerical study, possibly useful for further analytical investigation, is the antisymmetry property that implies that $\tan\theta$, and therefore A, in Eq. (38) is zero at the midpoint in time. This leads to an explicit relationship between Λ_1 and Λ_2:

$$\Lambda_2 = (1 - \Lambda_1) \tan(\tau_f/2)$$

The problem now reduces to a search for a one-parameter family of solutions which must be made in order to satisfy circular orbit terminal conditions. The resulting equations and integrals to be evaluated are still quite formidable and as yet have not yielded to analytical treatment.

Because the equations of motion are linear and contain only the single parameter P, it is possible to plot a "miles-per-gallon" nondimensional parameter, $y_f\omega_0^2/\tau_f(T/m)$, as a function of the nondimensional transfer time. This very general type of plot is presented in Fig. 7. Given the thrust/mass ratio of the vehicle and the frequency of the reference orbit, the increase in circular-orbit radius may be determined from Fig. 7 for any value of the orbital transfer angle.

It may be conjectured that the parameter P could be used as a basis of comparison for vehicles with different performance abilities and in dissimilar gravity environments, i.e., the ratio P is a measure of the relative influence of the thrust/force field.

Figure 8 shows the variation of Λ_1 and Λ_2 with τ_f up to four orbital periods. For increasing values of transfer time, the numerical values of Λ_1 and Λ_2 approach one and zero, respectively. The thrust program for these values, as may be expected, is circumferential.

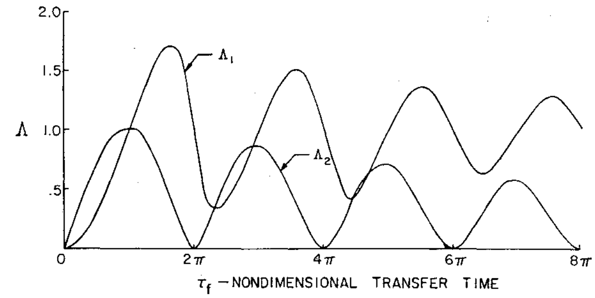

Fig. 8 Variation of Lagrange multipliers with nondimensional transfer time

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